To appear in the Proceedings of the 1996 IEEE International Conference on Robotics and Automation

Analysis of Probabilistic Roadmaps for Path Planning

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Abstract

We provide an analysis of a recent path planning method which uses probabilistic roadmaps. This method has proven very successful in practice, but the theoretical understanding of its performance is still limited. Assuming that a path γ exists between two configurations a and b of the robot, we study the dependence of the failure probability to connect a and b on (i) the length of γ, (ii) the distance function of γ from the obstacles, and (iii) the number of nodes N of the probabilistic roadmap constructed. Importantly, our results do not depend strongly on local irregularities of the configuration space, as was the case with previous analysis. These results are illustrated with a simple but illuminating example. In this example, we provide estimates for N, the principal parameter of the method, in order to achieve failure probability within prescribed bounds. We also compare, through this example, the different approaches to the analysis of the planning method.

1 Introduction

Motion planning has been an active area of research during the last two decades [12]. The problem has gained increasing attention because of the larger number of potential applications (e.g. robotics, manufacturing, computer-assisted surgery, molecular biology). Several recent papers describe practical path planners that can deal with robots that have more than 4 degrees of freedom (dof) and move in realistic environments (for a survey see [3, 7]). Because of the high computational complexity of path planning, these planners usually employ different heuristics to guide the search of the robot from its initial to its final position.

This paper considers the success of a class of probabilistic algorithms for path planning [6, 7, 8, 9, 13, 14] and tries to establish a framework for the theoretical understanding of their results. Our ultimate goal is to further enhance the performance of these methods by estimating good values for their input parameters. We will restrict ourselves to the description of the planner in [7, 8, 9] for a concise presentation of the algorithm and our results. We hereafter refer to this planner as `PRM` (Probabilistic RoadMap planner).

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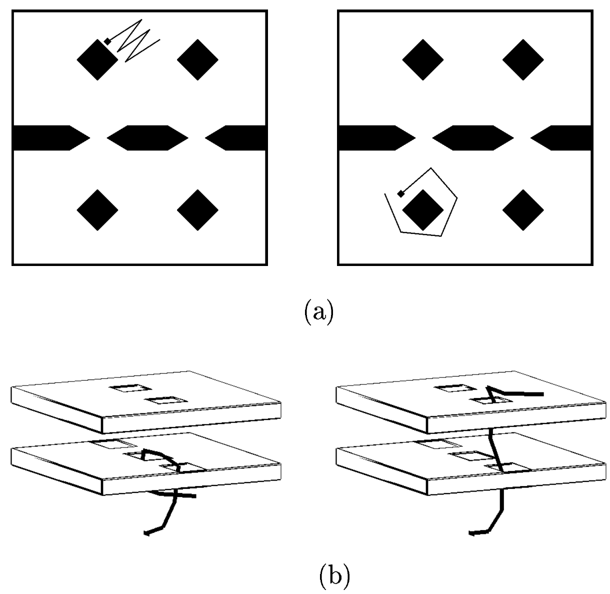

Figure 1: Examples of problems solved by `PRM`

`PRM` proceeds as follows. At a preprocessing stage, a probabilistic roadmap is constructed in the configuration space (C-space) of the robot. Initially, random configurations (nodes) are generated over the C-space of the robot and are interconnected with a deterministic and fast planner. We call this planner a connector to emphasize its simplicity (for example, the connector may examine only the straight-line path between two nodes). Each successful connection yields an edge of the roadmap. After a large number of nodes have been generated, the “difficult” (narrow) parts of the C-space are identified heuristically [7], and more nodes are placed in these areas. This faciliates the formation of roadmap components that correspond to the actual components of the free C-space. A path planning query specifies a and b, the initial and the final configurations of the robot. `PRM` connects them to nodes A and B of the same roadmap component using the connector, and then searches the roadmap for a sequence of edges from A to B. Concatenation of the relevant local paths produces an answer to the query. This path can be smoothed using any standard smoothing technique.

`PRM` has been applied with excellent results to free-flying and articulated robots moving in the plane or in

3-space, as well as to non-holonomic robots. Examples of its capabilities are given in Fig. 1. The robot in Fig. 1(a) has 7 dof. PRM answers path planning queries, like the one defined by the configurations in Fig. 1(a), in a fraction of a second after 50 seconds of preprocessing time on a DEC ALPHA workstation. For similar query times, 620 seconds are spent in the preprocessing stage for the robot of Fig. 1(b) which has 16 dof. Recently PRM has been applied to examples from assembly maintainability similar to the ones in [4] (aircraft engines).

As described in [7, 8, 9] PRM requires the tuning of several parameters which depend on the considered workspace and robot. For example, one such parameter is N, the size of the network that sufficiently captures the connectivity of the free C-space within a given probability. Currently, the output roadmap is augmented until the given initial and final configuration of the robot get connected through it. The theoretical estimation of N can make the full automation of the technique possible, and permit its application in a wide variety of environments with minimal user effort.

The theoretical analysis of PRM is a difficult task. The work in [10] initiated the analysis. It related the performance of the planner to the *goodness* of the C-space of the problem in consideration. A space S is called δ-good if the volume of S that each point in S can "see" is at least a δ fraction of the total free volume of S. In the PRM framework, a point sees another point if it can be connected to it by the connector. With the above definition, the value of δ is constrained by the point of the space which sees the least volume of S, which may be very small. Using δ we can estimate how many nodes a roadmap needs to have, so that the roadmap itself can see most of the C-space with high probability, and thus answer planning queries correctly with high probability.

The analysis in this paper focuses on understanding how the properties of the space in which the robot moves, the shape of the robot, and the features of the possible paths among distinct configurations influence parameters of the technique such as the size of the roadmaps that must be produced by preprocessing. We adopt the following point of view. Assuming that a path between two different configurations a and b of the robot exists, we show that the probability of failure to connect these configurations with PRM depends on (i) the length of the assumed path, (ii) the distance of the path from the obstacles, and (iii) the number of nodes of the roadmap generated. Using our results and making simple assumptions for the values of (i) and (ii) above, as well as for the failure probability we are willing to tolerate, we can estimate the size (number of nodes) of the probabilistic roadmap that finds a path between a and b with the given probability. Or, if the shortest path between a and b is known, we can estimate the size of the roadmap that will permit PRM to find a path which is η-close to the shortest path. Our analysis is not very sensitive to local difficulties of the C-space and carries over in any C-space dimension.

The analysis given in this paper together with the analysis in [10] are also presented in the context of the general planning scheme in [3]. In that work the distance of the robot from the obstacles in the workspace is used to define a random sampling scheme for path planning. PRM can be regarded as an instance the sampling scheme in [3]. Before proceeding let us also mention that the theoretical evaluation of algorithms that are experimentally successful in path planning has recently attracted considerable attention. Some examples of research in this direction can be found in [1, 5, 11].

2 Description of simplified PRM

To analyze the performance of PRM we work with a simplified algorithm, which we call the simplified Probabilistic Roadmap Planner (s-PRM). We rid the approach of the heuristics employed in practical implementations and any shortcuts taken to achieve better performance.

For the moment we assume that the C-space is two-dimensional. Later, we will show that our analysis can be carried over to higher C-space dimensions without any complications. The parameters of our model are:

- The Free Space
  An open subset $\mathcal{F}$ of the unit square $\mathcal{C} = [0, 1]^2$.
- The Robot
  A point which is free to move in $\mathcal{F}$.
- The Local Connector
  It takes the robot from point a to point b along a straight line and succeeds if the straight line segment $\overline{ab}$ is contained in $\mathcal{F}$.
- The Collection of Random Configurations
  A collection of N independent points uniformly distributed over $\mathcal{F}$.

s-PRM works as follows. We throw N independent random points in $\mathcal{F}$ and connect any two of them that can be connected by a free straight line. A roadmap G with possibly more than one connected components results in this fashion. To solve any planning problem, that is to go from any point a to any point b, we try to connect both a and b to two nodes in the same connected component of G using straight lines. s-PRM succeeds if and only if this is possible.

Our purpose is to analyze the probability of failure of s-PRM as a function of all the relevant parameters. For this we take any two points $a, b \in \mathcal{F}$, for which we assume that they can be connected via a rectifiable path

$$\gamma \ : \ [0, L] \longmapsto \mathcal{F}, \ \text{where } \gamma(0) = a \text{ and } \gamma(L) = b.$$

(The curve γ is parametrized by Euclidean arc length.) Let also $\mathcal{O}$ be the complement of $\mathcal{F}$ in $\mathcal{C}$ (the C-obstacle) and for any $x \in \mathcal{C}$ write $r(x)$ for the Euclidean distance of x to $\mathcal{O}$, that is

$$r(x) = \inf_{y \in \mathcal{O}} |x - y|,$$

where $|x - y|$ is the Euclidean distance of the points x and y of the plane.

We shall give upper bounds for the probability of failure of s-PRM to find a path from a to b. These bounds will involve the number N of random points, the function $r(\gamma(t))$ for $t \in [0, L]$, as well as the length L of γ, and will hold for any path γ that joins a and b. The dependence on L and $r(\gamma(t))$ is to be expected. If we have two points a and b for which any connecting path is long, it gets more difficult to find the path, since a larger number of relevant intermediate configurations must be present in our roadmap. Similarly, if any connecting path has small $r(\gamma(t))$, this intuitively means that the problem is difficult. Imagine for example that a and b are on different sides of narrow passage. The probability of placing random configurations inside the passage and connecting them by straight line paths is small.

3 A Bound that Involves the Minimum Distance from the Obstacles

In this section we derive an upper bound on the failure probability when connecting pairs of points a and b. It is assumed that a and b can be connected by *some* path

$$\gamma : [0, L] \longmapsto \mathcal{F},$$

which keeps uniformly away from the obstacles, that is all its points are at least a certain distance R away from the C-obstacles. The key idea is that of covering the path with few balls which overlap to a certain degree. (Here we follow closely [7].)

Theorem 1 *Let $\gamma : [0, L] \longmapsto \mathcal{F}$ be a path of (Euclidean) length L, with $\gamma(0) = a$, $\gamma(L) = b$, and let $R = \inf_{0 \le t \le L} r(\gamma(t))$ be the distance of the path to the obstacles.*

Then the probability that s-PRM *will fail to connect the points a and b is at most*

$$\frac{2L}{R}(1 - \alpha R^2)^N, \qquad (1)$$

where $\alpha = \pi/(4|\mathcal{F}|)$ is a constant.

Notation: We denote by $d(s,t)$ the distance of the points $\gamma(s)$ and $\gamma(t)$ *along* the curve γ. We also assume that γ is parametrized by arc length. The ball centered at $x \in \mathbb{R}^2$ and with radius r is denoted by $B_r(x)$.

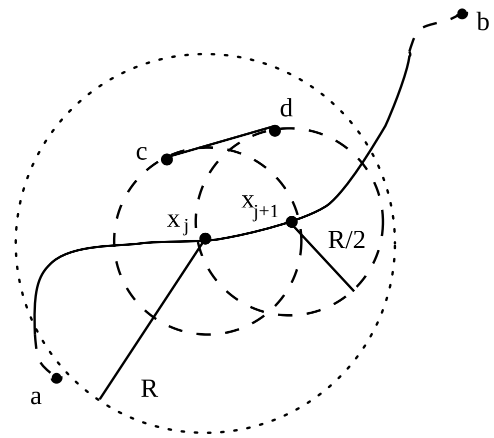

Figure 2: Proof of Theorem 1

Proof: Let $n = \lceil 2L/R \rceil$. Then we can find points $x_0 = a, x_1, \ldots, x_n = b$ on the curve γ, for which $d(x_j, x_{j+1}) \le R/2$, for all j. Notice then that

$$B_{R/2}(x_{j+1}) \subseteq B_R(x_j), \quad \text{for } j = 0, \ldots, n-1. \qquad (2)$$

This is a direct consequence of the triangle inequality and the inequality $|\gamma(s) - \gamma(t)| \le d(s,t)$.

Assume now that $c \in B_{R/2}(x_j)$ and $d \in B_{R/2}(x_{j+1})$. Observe then that also $d \in B_R(x_j)$ because of (2). This implies that the straight line segment $\overline{cd}$ is free, since both c and d are contained in the same free ball $B_R(x_j)$. An illustration of this basic fact is given in Fig. 2.

Let now $q_1, \ldots, q_N$ be the random points that our algorithm produced. According to the preceding observation, it is enough to have at least one of the q_k's, $k = 1, \ldots, N$, in each ball $B_{R/2}(x_j)$, $j = 1, \ldots, n-1$, for our algorithm to succeed to connect the points a and b. Since the q_k's are independent and uniformly distributed over $\mathcal{F}$, we conclude that the probability that the ball $B_{R/2}(x_j)$ contains none of the q_k's is equal to $(1 - |B_{R/2}|/|\mathcal{F}|)^N$, where $|B_{R/2}|$ is the area of the ball of radius $R/2$. Here we use the fact that we have thrown N *independent* points in $\mathcal{F}$. Thus,

$$\begin{aligned} \Pr[\text{FAILURE}] &\le \Pr[\text{Some ball is empty}] \\ &\le \sum_{j=1}^{n-1} \Pr[\text{The } j\text{-th ball is empty}] \\ &= \left(\left\lceil \frac{2L}{R} \right\rceil - 1\right)\left(1 - \frac{|B_{R/2}|}{|\mathcal{F}|}\right)^N \end{aligned} \qquad (3)$$

But since in two dimensions the area of the ball with radius $R/2$ is $\pi R^2/4$, the above relation becomes

$$\Pr[\text{FAILURE}] \le \frac{2L}{R}\left(1 - \frac{\pi R^2}{4|\mathcal{F}|}\right)^N,$$

which concludes the proof of the theorem. □

4 A Bound that Exploits Varying Distance

The analysis of Section 3 uses only the minimum distance of the path γ from the obstacles. Yet, if this minimum is achieved rarely, one expects the bound of Theorem 1 to be far from the truth. In this section we establish an upper bound for the failure probability that involves a "mean" distance from the obstacles. The idea of the proof is, as was the case in Section 3, to cover the curve γ with not-too-many balls that overlap to a certain extent.

Theorem 2 *Let the points $a, b \in \mathcal{F}$ be connected by a path $\gamma : [0, L] \to \mathcal{F}$, of Euclidean length L, and write*

$$r(t) = \inf_{x \in \mathcal{O}} |\gamma(t) - x|$$

for the distance of $\gamma(t)$ from the obstacles.

Then the probability of failure of `s-PRM` *is at most*

$$6 \int_0^L \frac{\left(1 - (\alpha/4) r^2(t)\right)^N}{r(t)} dt, \tag{4}$$

where α is again $\pi/(4|\mathcal{F}|)$.

Proof: Define $t_0 = 0$, $r_0 = r(0)$, and for $k \geq 0$ define

$$t_{k+1} = \sup\{t \in [t_k, L] : t - t_k \leq r_k - \frac{1}{2} r(t)\} \tag{5}$$

$$r_{k+1} = r(t_{k+1}), \tag{6}$$

and let n be defined by the requirement that $t_n = L$. We have so ensured that

$$B_{r_{k+1}/2}(\gamma(t_{k+1})) \subseteq B_{r_k}(\gamma(t_k)),$$

for $k = 0, \ldots, n-1$, and, by the same reasoning as in the proof of Theorem 1,

$$\Pr[\text{FAILURE}] \leq \sum_{k=1}^{n-1} (1 - \alpha r_k^2)^N. \tag{7}$$

Define the integral

$$I = \int_0^L \frac{(1 - (\alpha/4) r^2(t))^N}{r(t)} dt.$$

Let also

$$\overline{r}_k = \sup_{t_k \leq t \leq t_k + 1} r(t), \quad k = 0, \ldots, n-1.$$

The function $(1 - (\alpha/4) r^2)^N / r$ is a decreasing function of r. In each interval $[t_k, t_{k+1}]$, $k = 0, \ldots, n-1$, we estimate it from below by its infimum $(1 - (\alpha/4)\overline{r}_k^2)^N / \overline{r}_k$. We thus bound I from below with the corresponding lower Riemann sum

$$I \geq \sum_{k=0}^{n-1} (t_{k+1} - t_k) \frac{(1 - (\alpha/4)\overline{r}_k^2)^N}{\overline{r}_k}. \tag{8}$$

Claim 1: $\overline{r}_k \leq 2 r_k$.
This follows immediately from (5) since, if $\overline{r}_k = r(\overline{t}_k)$ for some $\overline{t}_k \in [t_k, t_{k+1}]$, we have

$$r_k - \frac{1}{2}\overline{r}_k \geq \overline{t}_k - t_k \geq 0.$$

Using Claim 1 equation (8) gives

$$I \geq \frac{1}{2} \sum_{k=0}^{n-1} \frac{t_{k+1} - t_k}{r_k} (1 - \alpha r_k^2)^N. \tag{9}$$

Claim 2: $\forall s, t: \ |r(s) - r(t)| \leq |s - t|$.
That is, the function $r(t)$ is *Lipschitz* with Lipschitz constant equal to 1. This is immediate from the triangle inequality (remember that $\gamma(t)$ has been parametrized by arc length) which gives

$$r(s) \leq |s - t| + r(t), \text{ and } r(t) \leq |s - t| + r(s).$$

Claim 3: $t_{k+1} - t_k \geq \frac{1}{3} r_k$, for $k = 0, \ldots, n-2$.
To see this notice that, by definition, we have

$$\begin{aligned} t_{k+1} - t_k &= r_k - \frac{1}{2} r_{k+1} \\ &\geq r_k - \frac{1}{2}\left(r_k + (t_{k+1} - t_k)\right) \text{ (Claim 2)} \\ &= \frac{1}{2} r_k - \frac{1}{2}(t_{k+1} - t_k), \end{aligned}$$

which implies $\frac{3}{2}(t_{k+1} - t_k) \geq \frac{1}{2} r_k$ and the claim. (Notice also that, if the event "the $(n-1)$-st ball is not empty" is relevant to the success of the algorithm, one has also $t_n - t_{n-1} \geq \frac{1}{2} r_{n-1} \geq \frac{1}{3} r_{n-1}$.)

Now (9), (7), and Claim 3 give

$$I \geq \frac{1}{6} \sum_{k=1}^{n-1} (1 - \alpha r_k^2)^N \geq \frac{1}{6} \Pr[\text{FAILURE}],$$

which concludes the proof of the theorem. □

5 Simplified Expressions

Using the inequality $1 - x \leq e^{-x}$, for $x \geq 0$, we get, from Theorems 1 and 2, the following easier-to-use upper bounds for the failure probability.

- The bound of Theorem 1 becomes

$$\Pr[\text{FAILURE}] \leq \frac{2L}{R} \exp(-\alpha R^2 N). \tag{10}$$

- The bound of Theorem 2 becomes

$$\Pr[\text{FAILURE}] \leq 6 \int_0^L \frac{\exp(-(\alpha/4) r^2(t) N)}{r(t)} dt, \tag{11}$$

In both formulas above $\alpha = \pi/(4|\mathcal{F}|)$.

6 Analysis of a Particular Problem

We have found simple upper bounds for the probability of failure to find a given path with the probabilistic roadmaps method. We are now going to use these bounds to derive estimates on N, the number of random points thrown uniformly in the free C-space, in order to have the probability of failure less than a prespecified number, say, for the sake of the argument, less than $1/2$.

In this section we study a simple problem in two dimensions. For the problem shown in Fig. 3 we estimate N using equation (10), equation (11) and the method of analysis of [10].

The parameter of the problem is ϵ, the length of the opening near point a, which is taken to tend to 0.

We have $|\mathcal{F}| = 1$ so that $\alpha = \pi/4$. We also have $L \asymp 1$ and $R \asymp \epsilon$. (By $x \asymp y$ we mean that $C^{-1}x \le y \le Cy$, for some absolute constant $C > 0$. In what follows C stands for an absolute positive constant, not necessarily the same in all its occurrences.)

Estimate using (10):
We get

$$\begin{aligned} \Pr[\text{FAILURE}] &\le \frac{2L}{R}\exp(-\alpha R^2 N) \\ &\le C\frac{1}{\epsilon}\exp(-C\epsilon^2 N). \end{aligned}$$

If we choose

$$N \asymp \frac{1}{\epsilon^2}\log\frac{1}{\epsilon} \tag{12}$$

we achieve that the failure probability is bounded above by a small constant (which, of course, depends on the constant implied by the $\asymp$ sign in (12), but it is the dependence of N on ϵ that we care about here).

Estimate using (11):
The function $r(t)$ for the path of Fig. 3 clearly satisfies

$$r(t) \asymp \begin{cases} \epsilon & \text{if } 0 \le t \le C\epsilon \\ t & \text{if } t \ge C\epsilon \end{cases}.$$

Equation (11) then gives

$$\begin{aligned} \Pr[\text{FAILURE}] &\le 6\int_0^L \frac{\exp(-(\alpha/4)r^2(t)N)}{r(t)}dt \\ &\le C\exp(-C\epsilon^2 N)\int_0^L \frac{dt}{r(t)} \end{aligned}$$

and

$$\begin{aligned} \int_0^L \frac{dt}{r(t)} &\le \int_0^{C\epsilon}\frac{dt}{r(t)} + C\int_{C\epsilon}^L \frac{dt}{t} \\ &\asymp C\epsilon\frac{1}{\epsilon} + \log\frac{1}{\epsilon} \asymp \log\frac{1}{\epsilon}. \end{aligned}$$

Therefore

$$\Pr[\text{FAILURE}] \le C\exp(-C\epsilon^2 N)\log\frac{1}{\epsilon},$$

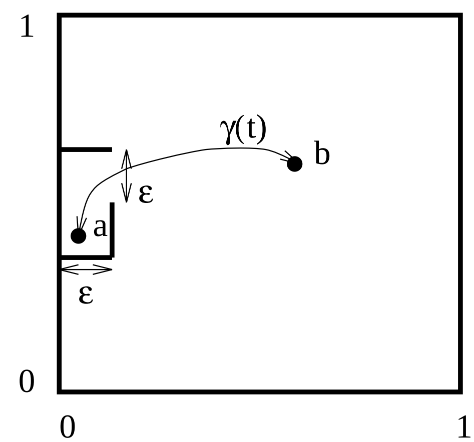

Figure 3: A particular problem

and choosing

$$N \asymp \frac{1}{\epsilon^2}\log\log\frac{1}{\epsilon} \tag{13}$$

bounds the failure probability from above by a small constant.

Estimate after [10]:
The space in Fig. 3 is, in the terminology of [10], a $(C\epsilon^2)$-good space. This means that every point of $\mathcal{F}$ can be connected with a free straight line segment to a set of points of $\mathcal{F}$ whose area is at least $C\epsilon^2$ (clearly because of the box on the left).

Then (see Theorem 2.1 in [10] and the definition of *adequate* sets of points) one needs to have

$$N \asymp \frac{1}{\epsilon^2}\log\frac{1}{\epsilon} \tag{14}$$

in order to bound the failure probability away from 1.

Comparison:
Theorem 2 clearly exploits the fact that $r(t)$ is small only briefly to gain an extra logarithm in the estimate (13) with respect to (12) and (14). Note that Theorem 1 and the approach taken in [10] refer to quantities – the minimum of $r(t)$ and the *goodness* of $\mathcal{F}$ – which are single numbers defined globally over the whole space. Theorem 2 achieves a better estimate for N because it is less sensitive to local difficulties than are the bounds of Theorem 1 and the analysis in [10].

It should be said that the true answer is

$$N \asymp \frac{1}{\epsilon^2}, \tag{15}$$

since it is necessary and sufficient to put a bounded number of points in the box, which happens with probability $\asymp \epsilon^2$. The estimates (12), (13) and (14) can thus be seen as the unavoidable $1/\epsilon^2$ times a factor on which they are to be compared. According to (11) and after estimating (from above) the exponential by

$\exp(-(\alpha/2)R^2N)$ (remember $R = \inf_t r(t)$) that factor takes the very simple form

$$\frac{L}{R_h} = \int_0^L \frac{dt}{r(t)},$$

where R_h is the *harmonic mean* of the function $r(t)$.

7 Higher Dimension

Everything in the preceding analysis extends without any changes to spaces of higher dimension. Let d be the dimension of the space and ω_d denote the d-dimensional volume of the unit ball in $\mathbb{R}^d$. Let also $\alpha_d = 2^{-d}\omega_d/|\mathcal{F}|$.

- The bound corresponding to (10) becomes then

$$\Pr[\text{FAILURE}] \le \frac{2L}{R}\exp(-\alpha_d R^d N). \qquad (16)$$

- The bound corresponding to (11) becomes

$$\Pr[\text{FAILURE}] \le 6\int_0^L \frac{\exp(-\alpha_d 2^{-d} r^d(t) N)}{r(t)} dt. \qquad (17)$$

8 Discussion

The bounds computed in this paper are not very easy to use since they depend on the properties of the postulated connecting path $\gamma(t)$ from a to b, which are difficult to measure *a priori*. Nevertheless, they at least shed light on the nature of the dependence of the algorithm on these properties. The fact that the dependence on N is exponential is a good and, of course, expected feature. Another nice feature revealed is that the dependence on L is linear. The bound of Theorem 2 that exploits varying distance from the obstacles makes our analysis useful in spaces where there are narrow regions and large parts of free C-space.

What the bounds given by Theorems 1 and 2 and by the simplified inequalities (10) and (11) do permit us to do is to answer questions of the type: *" Assuming that there is a path from a to b which stays away from the obstacles a distance at least ϵ, what should N be in order to guarantee a success probability of at least* 0.99*?"* Or, if we know the optimal path from a to b and this stays ϵ-away from the obstacles, we can estimate what N should be in order to find, with a predefined probability, a path which stays η-close ($\eta < \epsilon$) to the optimal path. This is done simply by using η as the distance of the optimal path from the obstacles.

We have thus obtained quite workable expressions for the failure probability and have demonstrated their use with a simple but illuminating example. It should also be said that, in practice, one need not restrict oneself to using the Euclidean distance. No special properties of it were used in this paper and any other distance would give analogous results.

The analysis in this paper relates in a direct way geometric properties of the configuration space of the robot to the parameters of the `PRM` planning algorithm. We hope that research along the direction of understanding how the geometric properties of the robot's environment influence the performance of specific planning algorithms will guide us in the design of better planners.

Acknowledgments: L. Kavraki and J.-C. Latombe were partially supported by ARO MURI grant DAAH04-96-1-007. M. Kolountzakis was supported by NSF grant DMS-9304580.

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